ON A POLAR REPRESENTATION OF NON-SINGULAR SQUARE MATRICES

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Let $\mathbf{A}$ be a square matrix of n rows and let the element in the p^{th} row and q^{th} column of $\mathbf{A}$ be denoted by a_{pq}; defining the square matrix $\mathbf{A}^*$ by the equations $a^*_{pq} = \bar{a}_{qp}$ (where the superposed bar indicates *conjugate complex*) we may construct the two *norms* $\mathbf{N}_1 = \mathbf{AA}^*$ and $\mathbf{N}_2 = \mathbf{A}^*\mathbf{A}$ of $\mathbf{A}$. These norms are in general different but when they are equal the matrix $\mathbf{A}$ is said to be normal. We shall consider only the first norm $\mathbf{N}_1$ in what follows it being clear that our remarks apply equally to $\mathbf{N}_2$.

It is evident that $\mathbf{N}_1 = \mathbf{N}_1^*$, i.e., that $\mathbf{N}_1$ is Hermitian. With any Hermitian matrix $\mathbf{H}$ may be associated a *composite form* $\phi(\mathbf{H}) \equiv \sum\limits_{pq} h_{pq}x_p\bar{x}_q$ assuming real values. The form $\phi(\mathbf{N}_1) = \sum\limits_s(\sum\limits_p a_{ps}x_p)(\sum\limits_p \bar{a}_{ps}\bar{x}_p)$ cannot take negative values and takes the value zero only when all the sums $\sum\limits_p a_{ps}x_p$ are zero. If, as we shall suppose, $\mathbf{A}$ is non-singular this means that $\phi(\mathbf{N}_1)$ takes the value zero only when all the x's are zero, assuming positive values otherwise. In other words $\mathbf{N}_1$ is positively definite. If new variables y are introduced by the formulae $x_p = \sum\limits_s u_{sp}y_s$ where $\mathbf{U}$ is any unitary matrix ($\mathbf{UU}^* = \mathbf{E}$, the unit matrix) $\phi(\mathbf{N}_1)$ is transformed into a composite form whose associated Hermitian matrix is $\mathbf{UN}_1\mathbf{U}^*$ and accordingly this matrix is positively definite ($\mathbf{U}$ being necessarily non-singular). Now the Hermitian matrix $\mathbf{N}_1$ may be transformed[1] by a unitary matrix into a diagonal matrix $[\mathbf{N}_1] = \mathbf{UN}_1\mathbf{U}^*$ and since $[\mathbf{N}_1]$ is positively definite its diagonal elements λ_p (which are the characteristic numbers of $\mathbf{N}_1$) are positive. Hence the diagonal matrix $[\mathbf{P}_1]$ whose diagonal elements are $+\sqrt{\lambda_p}$ is positively definite and has its square $=$ $[\mathbf{N}_1]$. This implies that the matrix $\mathbf{P}_1 = \mathbf{U}^*[\mathbf{P}_1]\mathbf{U}$ is positively definite and has its square $= \mathbf{N}_1$. That there exists no other positively definite matrix $\mathbf{Q}$ whose square is $\mathbf{N}_1$ is shown as follows. Any such matrix $\mathbf{Q}$ would have for its characteristic numbers the positive square roots of the characteristic numbers of $\mathbf{N}_1$ and would, accordingly, be unitarily equivalent to $\mathbf{P}_1$ (both matrices having the same characteristic numbers):

$$\mathbf{Q} = \mathbf{VP}_1\mathbf{V}^*; \quad \mathbf{VV}^* = \mathbf{E}.$$

This would imply $\mathbf{N}_1 = \mathbf{VN}_1\mathbf{V}^*$ or, equivalently,

$$[\mathbf{N}_1] = \mathbf{W}[\mathbf{N}_1]\mathbf{W}^*; \quad \mathbf{W} = \mathbf{UVU}^*$$

Since $[\mathbf{N}_1]$ is a diagonal matrix it readily follows by direct computation of $[\mathbf{N}_1]\mathbf{W}$ and $\mathbf{W}[\mathbf{N}_1]$ that if all the characteristic numbers of $\mathbf{N}_1$ are different $\mathbf{W}$ is a diagonal matrix and this implies $\mathbf{Q} = \mathbf{P}_1$. For then $\mathbf{W}[\mathbf{P}_1] = [\mathbf{P}_1]\mathbf{W}$ and so $\mathbf{Q} = \mathbf{V}\mathbf{P}_1\mathbf{V}^* = \mathbf{U}^*\mathbf{W}\mathbf{U}\mathbf{P}_1\mathbf{U}^*\mathbf{W}^*\mathbf{U} = \mathbf{U}^*\mathbf{W}[\mathbf{P}_1]\text{-}\mathbf{W}^*\mathbf{U} = \mathbf{U}^*[\mathbf{P}_1]\mathbf{U} = \mathbf{P}_1$. If several of the characteristic numbers of $\mathbf{N}_1$ are equal $\mathbf{W}$ need not be a diagonal matrix but will nevertheless be such that $\mathbf{W}[\mathbf{P}_1]\mathbf{W}^* = [\mathbf{P}_1]$ since to each equality among the characteristic numbers of $\mathbf{N}_1$ there is a corresponding equality among the characteristic numbers of $\mathbf{P}_1$.[2] The fact that $\mathbf{Q} = \mathbf{P}_1$ follows by the argument just given. The positively definite matrix thus determined uniquely has a reciprocal $\mathbf{P}_1^{-1}$ and it is clear that the matrix $\mathbf{U} = \mathbf{P}_1^{-1}\mathbf{A}$ is unitary; for $\mathbf{U}\mathbf{U}^* = \mathbf{P}_1^{-1}\mathbf{A}\mathbf{A}^*\mathbf{P}_1^{-1} = \mathbf{E}$. We have, then, the representation $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ of the non-singular matrix $\mathbf{A}$ as the product of a positively definite matrix $\mathbf{P}_1$ by a unitary matrix and the representation is *unique;* for from $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ follows $\mathbf{A}\mathbf{A}^* = \mathbf{P}_1^2$ which (together with the positive definite character of $\mathbf{P}_1$) determines $\mathbf{P}_1$ uniquely. $\mathbf{P}_1$ being determined $\mathbf{U}$ follows unambiguously as $\mathbf{P}_1^{-1}\mathbf{A}$.

Using the second norm we can represent $\mathbf{A}$ unambiguously in the form $\mathbf{V}\mathbf{P}_2$ and it is immediately evident that $\mathbf{V} = \mathbf{U}$; for $\mathbf{A} = \mathbf{V}\mathbf{P}_2 = \mathbf{V}\mathbf{P}_2\mathbf{V}^*.\mathbf{V}$ and $\mathbf{V}\mathbf{P}_2\mathbf{V}^*$ being positively definite it follows from the proven uniqueness of the analysis of $\mathbf{A}$ into the form $\mathbf{P}_1\mathbf{U}$ that $\mathbf{V}\mathbf{P}_2\mathbf{V}^* = \mathbf{P}_1$; $\mathbf{V} = \mathbf{U}$. We state then the following theorem:

Any non-singular square matrix $\mathbf{A}$ *may be represented uniquely in the forms*

$$\mathbf{A} = \mathbf{P}_1\mathbf{U} = \mathbf{U}\mathbf{P}_2,$$

where $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are positively definite and* $\mathbf{U}$ *is unitary. Either of these representations we call a polar representation* (for $n = 1$, $\mathbf{A}$ is an ordinary complex number $a = re^{i\theta}$ and $\mathbf{P}_1$ is the modulus r while $\mathbf{U}$ is the turn $e^{i\theta}$).

The characteristic numbers λ of the matrix $\mathbf{A}$ are invariants and *a fortiori* unitary invariants. There are, however, other unitary invariants. The problem of determining the complete system of unitary invariants of an arbitrary square matrix seems to be as yet unsolved[3] but we may state the following theorem:

The characteristic numbers of the "polar coördinates" $\mathbf{P}_1$ and $\mathbf{U}$ of $\mathbf{A}$ are unitary invariants of $\mathbf{A}$.

In fact, from $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ follows $\mathbf{V}\mathbf{A}\mathbf{V}^* = \mathbf{V}\mathbf{P}_1\mathbf{V}^*.\mathbf{V}\mathbf{U}\mathbf{V}^*$ ($\mathbf{V}$ being a unitary matrix) and $\mathbf{V}\mathbf{P}\mathbf{V}^*$ being positively definite with $\mathbf{P}_1$ it follows that $\mathbf{V}\mathbf{P}_1\mathbf{V}^*$ and $\mathbf{V}\mathbf{U}\mathbf{V}^*$ are the polar coördinates of $\mathbf{V}\mathbf{A}\mathbf{V}^*$. Hence any unitary invariant of either $\mathbf{P}_1$ or $\mathbf{U}$ is a unitary invariant of $\mathbf{A}$; in fact if $\phi(\mathbf{P}_1) = F(\mathbf{A})$ is a unitary invariant of $\mathbf{A}$ we have $F(\mathbf{V}\mathbf{A}\mathbf{V}^*) = \phi(\mathbf{V}\mathbf{P}_1\mathbf{V}^*) = \phi(\mathbf{P}_1) = F(\mathbf{A})$ and similarly for $\mathbf{U}$. In particular the symmetric functions of the characteristic numbers of $\mathbf{P}_1$ and $\mathbf{U}$ are unitary invariants of $\mathbf{A}$.

Included among these is the sum of the squares of the characteristic numbers of $\mathbf{P}_1$, i.e., the sum of the characteristic numbers of $\mathbf{N}_1 = \mathbf{AA}^*$; this is the well-known unitary invariant $\sum\limits_{p,q} a_{pq}\bar{a}_{pq}$ of Frobenius.

When $\mathbf{A}$ is normal $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$ or $\mathbf{P}_1\mathbf{U}.\mathbf{U}^*\mathbf{P}_1 = \mathbf{U}^*{}_1\mathbf{P}.\mathbf{P}_1\mathbf{U}$ so that $\mathbf{P}_1^2 = \mathbf{U}^*\mathbf{P}_1^2\mathbf{U} = (\mathbf{U}^*\mathbf{P}_1\mathbf{U})^2$. Hence $\mathbf{P}_1 = \mathbf{U}^*\mathbf{P}_1\mathbf{U}$ or $\mathbf{UP}_1 = \mathbf{P}_1\mathbf{U}$. Conversely if $\mathbf{UP}_1 = \mathbf{P}_1\mathbf{U}$ we have $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{A}$ so that a *matrix* $\mathbf{A}$ *is normal when and only when its polar coördinates are commutable, that is, when the polar representations* $\mathbf{A} = \mathbf{P}_1\mathbf{U}$, $\mathbf{A} = \mathbf{UP}_2$ *coincide.*

It may be mentioned that the above considerations are valid also in the real domain. In this case the polar representation is simply the algebraic formulation of the fact, well known for $n = 3$ from the kinematics of homogeneous linear (non-singular) deformations, that any such deformation may be represented as a superposition of a dilatation and a rotation (the norm $\mathbf{AA}^*$ of $\mathbf{A}$ determining the ellipsoid of dilatation belonging to the deformation $\mathbf{A}$).

[1] See, for example, Weyl, H.; *Gruppentheorie und Quantenmechanik,* Leipzig, pp. 19–23, 1928.

[2] See Weyl, H., loc. cit.

[3] Since writing the above this problem has been solved and will be treated in a forthcoming note in these PROCEEDINGS.

NOTE ON THE HEAVISIDE EXPANSION FORMULA

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The Expansion Formula solution of the linear differential equation with constant coefficients

$$a_0 \frac{d^n x}{dyn} + a_1 \frac{d^{n-1}x}{dy^{n-1}} + \ldots + a_n y = F$$

was first stated by Oliver Heaviside. Perhaps due to his rather obscure methods of presentation, various writers have stated that the formula was given without proof. Nevertheless, Heaviside gave two proofs of the formula which may be traced through his writings. One of these, which we may designate as the second of Heaviside's proofs, was discussed a few years ago by Vallarta.[1] Heaviside really made no clear point of demarcation between his proofs and in all probability did not believe any proof was necessary. The Expansion Formula was but a single result of his devious analyses in the solution of certain differential